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A connection between the Lorentz condition and the Bianchi identity

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Abstract. We show for totally antisymmetric tensor fields of rank 1, 2 or 3, describing massive spin-one or spin-zero fields, that there is a simple connection between the operators in the Lagrange densities that yield either the Lorentz condition or the Bianchi identity. These operators are, up to a factor of $1/m^2$, each other's Klein-Gordon divisors.

Consider a general, massive tensor field ϕ of rank r and let $I^{(r)}$ be the identity operator so that

$$\phi = I^{(r)}\phi. \tag{1}$$

Let

$$\mathscr{L} = \phi^* \Lambda^{(r)} \phi \tag{2}$$

be the Lagrange density for such a free rank-r tensor field. Basically $\Lambda^{(r)}$ is the differential operator in the Euler-Lagrange equations for ϕ. The Klein-Gordon divisor $D^{(r)}$ is defined by

$$\Lambda^{(r)}D^{(r)} = D^{(r)}\Lambda^{(r)} = -(\partial^2 + m^2)I^{(r)}. \tag{3}$$

This reciprocal relationship between $\Lambda^{(r)}$ and $D^{(r)}$ strongly suggests that if (2) is the Lagrange density for fields of definite spin then

$$\mathscr{L}' = m^2 \phi^* D^{(r)} \phi \tag{4}$$

should also be a Lagrange density for fields of definite spin. Such a reciprocal relationship is already implicit in the work of Aurilia and Umezawa [1].

The purpose of this paper is to explicitly exhibit such a reciprocal relationship for massive, totally antisymmetric tensor fields of rank r ($r = 1, 2, 3$).

Totally antisymmetric tensor fields of rank r ($r = 1, 2, 3$) have been used to describe massive particles of spin-zero and spin-one [2]. These fields can either satisfy a Lorentz condition or a Bianchi identity to restrict the degrees of freedom (both conditions are Lorentz covariant). We present a simple equivalence theorem between these two conditions (the Lorentz condition as opposed to the Bianchi identity).

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In this case let ϕ be a rank-r ($r = 1, 2, 3$) totally antisymmetric tensor field and let $d^{(r)}$ be the projection operator that projects out the totally antisymmetric part from an arbitrary tensor field of rank r. Then $d^{(r)}$ plays the role of the identity operator $I^{(r)}$ when operating on totally antisymmetric tensor fields. Thus

$$\phi \equiv d^{(r)}\phi. \tag{5}$$

The various $d^{(r)}$ are given below. Now again let

$$\mathscr{L} = \phi^* \Lambda \phi \tag{6}$$

be the Lagrange density for the free rank-r tensor field. Then we have two possible Λ, namely a $\Lambda^{(\mathrm{L})}$ which yields the Lorentz condition and a $\Lambda^{(\mathrm{B})}$ which yields the Bianchi identity [3]. In each case we write out both of these conditions.

The result we have discovered is that $\Lambda^{(\mathrm{L})}$ and $\Lambda^{(\mathrm{B})}$ are proportional to each other's Klein-Gordon divisor. Thus

$$\frac{1}{m^2}\Lambda^{(\mathrm{L})}\Lambda^{(\mathrm{B})} = \frac{1}{m^2}\Lambda^{(\mathrm{B})}\Lambda^{(\mathrm{L})} = -(\partial^2 + m^2)d^{(r)}. \tag{7}$$

We now present this result explicitly for the three cases of $r = 1$, 2 and 3.

(*i*) $r = 1$

In this case

$$\phi_\mu = d^{(1)}_{\mu:\rho}\phi^\rho \qquad \text{with } d^{(1)}_{\mu:\rho} = g_{\mu\rho} \tag{8}$$

and

$$\Lambda^{(\mathrm{B})}_{\mu:\rho}(\partial) = m^2 P^{(1)}_{\mu:\rho} \tag{9}$$

$$\Lambda^{(\mathrm{L})}_{\mu:\rho}(\partial) = m^2 P^{(1)}_{\mu:\rho} - m^2 d^{(1)}_{\mu:\rho} - (\partial^2 + m^2) d^{(1)}_{\mu:\rho} \tag{10}$$

where

$$P^{(1)}_{\mu:\rho} \equiv g_{\mu\rho} + \frac{1}{m^2}\partial_\mu\,\partial_\rho = d^{(1)}_{\mu:\rho} + \frac{1}{m^2} d^{(1)}_{\mu:\alpha}\,\partial^\alpha\,\partial_\beta d^{(1)\beta}{}_{:\rho}. \tag{11}$$

A useful relation for this operator is

$$P^{(1)}_{\mu:\alpha}P^{(1)\alpha}{}_{:\rho} = P^{(1)}_{\mu:\rho} + \frac{1}{m^2}(\partial^2 + m^2)[P^{(1)}_{\mu:\rho} - d^{(1)}_{\mu:\rho}]. \tag{12}$$

Using this expression given by equations (9) and (10) as well as the result in equation (12) we find that

$$\frac{1}{m^2}\Lambda^{(\mathrm{L})}_{\mu:\alpha}(\partial)\Lambda^{(B)\alpha}{}_{:\rho}(\partial) = \frac{1}{m^2}\Lambda^{(\mathrm{B})}_{\mu:\alpha}(\partial)\Lambda^{(\mathrm{L})\alpha}{}_{:\rho}(\partial) = -(\partial^2 + m^2)d^{(1)}_{\mu:\rho} \tag{13}$$

as required.

By way of reference we simply state that the Lagrange density with $\Lambda^{(\mathrm{L})}$ yields the Lorentz condition

$$\partial^\mu\phi_\mu = 0 \tag{14}$$

whereas the Lagrange density with $\Lambda^{(\mathrm{B})}$ yields the Bianchi identity

$$\partial_\lambda\phi_\mu - \partial_\mu\phi_\lambda = 0. \tag{15}$$

(*ii*) $r=2$
Here we have

$$\phi_{\mu\nu} = d^{(2)}_{\mu\nu:\rho\sigma}\phi^{\rho\sigma} \qquad \text{with} \qquad d^{(2)}_{\mu\nu:\rho\sigma} = \tfrac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{16}$$

Also

$$\Lambda^{(\mathrm{B})}_{\mu\nu:\rho\sigma}(\partial) = m^2 P^{(2)}_{\mu\nu:\rho\sigma} \tag{17}$$

$$\Lambda^{(\mathrm{L})}_{\mu\nu:\rho\sigma}(\partial) = m^2 P^{(2)}_{\mu\nu:\rho\sigma} - m^2 d^{(2)}_{\mu\nu:\rho\sigma} - (\partial^2 + m^2) d^{(2)}_{\mu\nu:\rho\sigma} \tag{18}$$

where

$$P^{(2)}_{\mu\nu:\rho\sigma} = \tfrac{1}{2}[P^{(1)}_{\mu:\rho}P^{(1)}_{\nu:\sigma} - P^{(1)}_{\mu:\sigma}P^{(1)}_{\nu:\rho}]. \tag{19}$$

This expression can also be rewritten in terms of $d^{(2)}$ to yield

$$P^{(2)}_{\mu\nu:\rho\sigma} = d^{(2)}_{\mu\nu:\rho\sigma} + \frac{2}{m^2} d^{(2)}_{\mu\nu:\alpha\beta}\,\partial^\alpha\,\partial_\gamma d^{(2)\gamma\beta}{}_{:\rho\sigma} \tag{20}$$

and then we find the useful relationship

$$P^{(2)}_{\mu\nu:\alpha\beta}P^{(2)\alpha\beta}{}_{:\rho\sigma} = P^{(2)}_{\mu\nu:\rho\sigma} + \frac{1}{m^2}(\partial^2 + m^2)[P^{(2)}_{\mu\nu:\rho\sigma} - d^{(2)}_{\mu\nu:\rho\sigma}]. \tag{21}$$

A straightforward computation using these definitions and result yields

$$\frac{1}{m^2}\Lambda^{(\mathrm{L})}_{\mu\nu:\alpha\beta}(\partial)\Lambda^{(\mathrm{B})\alpha\beta}{}_{:\rho\sigma}(\partial) = \frac{1}{m^2}\Lambda^{(\mathrm{B})}_{\mu\nu:\alpha\beta}(\partial)\Lambda^{(\mathrm{L})\alpha\beta}{}_{:\rho\sigma}(\partial)$$
$$= -(\partial^2 + m^2) d^{(2)}_{\mu\nu:\rho\sigma} \tag{22}$$

as stated.

The corresponding Lorentz condition and Bianchi identity are respectively given by

$$\partial^\mu \phi_{\mu\nu} = 0 \tag{23}$$

$$\partial_\lambda \phi_{\mu\nu} + \partial_\mu \phi_{\nu\lambda} + \partial_\nu \phi_{\lambda\mu} = 0. \tag{24}$$

(*iii*) $r=3$
The antisymmetry of the field is specified by

$$\phi_{\mu\nu\lambda} = d^{(3)}_{\mu\nu\lambda:\rho\sigma\kappa}\phi^{\rho\sigma\kappa} \tag{25}$$

with

$$d^{(3)}_{\mu\nu\lambda:\rho\sigma\kappa} = \tfrac{1}{6}(g_{\mu\rho}g_{\nu\sigma}g_{\lambda\kappa} + g_{\mu\sigma}g_{\nu\kappa}g_{\lambda\rho} + g_{\mu\kappa}g_{\nu\rho}g_{\lambda\sigma} - g_{\mu\rho}g_{\nu\kappa}g_{\lambda\sigma} - g_{\mu\kappa}g_{\nu\sigma}g_{\lambda\rho} - g_{\mu\sigma}g_{\nu\rho}g_{\lambda\kappa}). \tag{26}$$

The operators in the Lagrange densities are

$$\Lambda^{(\mathrm{B})}_{\mu\nu\lambda:\rho\sigma\kappa}(\partial) = m^2 P^{(3)}_{\mu\nu\lambda:\rho\sigma\kappa} \tag{27}$$

$$\Lambda^{(\mathrm{L})}_{\mu\nu\lambda:\rho\sigma\kappa}(\partial) = m^2 P^{(3)}_{\mu\nu\lambda:\rho\sigma\kappa} - m^2 d^{(3)}_{\mu\nu\lambda:\rho\sigma\kappa} - (\partial^2 + m^2) d^{(3)}_{\mu\nu\lambda:\rho\sigma\kappa} \tag{28}$$

where

$$P^{(3)}_{\mu\nu\lambda:\rho\sigma\kappa} = \tfrac{1}{6}[P^{(1)}_{\mu:\rho}P^{(1)}_{\nu:\sigma}P^{(1)}_{\lambda:\kappa} + P^{(1)}_{\mu:\sigma}P^{(1)}_{\nu:\kappa}P^{(1)}_{\lambda:\rho} + P^{(1)}_{\mu:\kappa}P^{(1)}_{\nu:\rho}P^{(1)}_{\lambda:\sigma}$$
$$- P^{(1)}_{\mu:\rho}P^{(1)}_{\nu:\kappa}P^{(1)}_{\lambda:\sigma} - P^{(1)}_{\mu:\kappa}P^{(1)}_{\nu:\sigma}P^{(1)}_{\lambda:\rho} - P^{(1)}_{\mu:\sigma}P^{(1)}_{\nu:\rho}P^{(1)}_{\lambda:\kappa}]. \tag{29}$$

After a rather tedious computation this can also be rewritten in terms of $d^{(3)}$ to yield

$$P^{(3)}_{\mu\nu\lambda:\rho\sigma\kappa} = d^{(3)}_{\mu\nu\lambda:\rho\sigma\kappa} + \frac{3}{m^2} d^{(3)}_{\mu\nu\lambda:\alpha\beta\gamma} \partial^\alpha \partial_\delta d^{(3)\delta\beta\gamma}{}_{:\rho\sigma\kappa}. \tag{30}$$

One then further finds the relationship

$$P^{(3)}_{\mu\nu\lambda:\alpha\beta\gamma} P^{(3)\alpha\beta\gamma}{}_{:\rho\sigma\kappa} = P^{(3)}_{\mu\nu\lambda:\rho\sigma\kappa} + \frac{1}{m^2}(\partial^2 + m^2)[P^{(3)}_{\mu\nu\lambda:\rho\sigma\kappa} - d^{(3)}_{\mu\nu\lambda:\rho\sigma\kappa}]. \tag{31}$$

In a similar manner as before we apply these results to find

$$\begin{aligned} \frac{1}{m^2} \Lambda^{(L)}_{\mu\nu\lambda:\alpha\beta\gamma} \Lambda^{(B)\alpha\beta\gamma}{}_{:\rho\sigma\kappa} &= \frac{1}{m^2} \Lambda^{(B)}_{\mu\nu\lambda:\alpha\beta\gamma} \Lambda^{(L)\alpha\beta\gamma}{}_{:\rho\sigma\kappa} \\ &= -(\partial^2 + m^2) d^{(3)}_{\mu\nu\lambda:\rho\sigma\kappa}. \end{aligned} \tag{32}$$

This concludes the demonstration of our result. The Lorentz condition and Bianchi identity for this case are respectively given by

$$\partial^\mu \phi_{\mu\nu\lambda} = 0 \tag{33}$$

$$\partial_\kappa \phi_{\mu\nu\lambda} - \partial_\mu \phi_{\nu\lambda\kappa} + \partial_\nu \phi_{\lambda\kappa\mu} - \partial_\lambda \phi_{\kappa\mu\nu} = 0. \tag{34}$$

The spin content of these tensor fields is easily obtained by just counting their degrees of freedom. This follows from the fact that they are massive and that, therefore, the little group for the Poincaré group is just O(3). Such a counting procedure yields the following results:

Lorentz condition: the spins are 1, 1 and 0 corresponding, respectively, to $r = 1$, 2 and 3;

Bianchi identity: the spins are 0, 1 and 1 corresponding, respectively, to $r = 1$, 2 and 3.

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